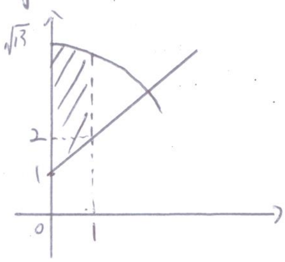
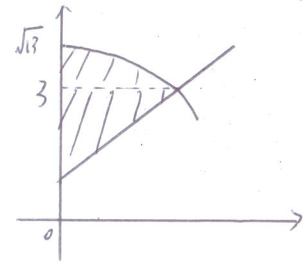
Suggested solutions of midterm

Q1. (a) False.

Reason: Left-hand side:



Right-hand side:



(b) True.

(c) False.

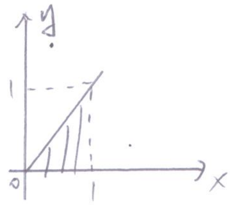
Reason: This is not symmetry from top to bottom.

(d) False.

Reason: Should be $\int_C F\,ds = f(P_1) - f(P_0)$.

(e) True.

Q2. Solution:



$$\int_0^1 \int_0^x \frac{\sin(1-y)}{1-y}\,dy\,dx = \int_0^1 \int_y^1 \frac{\sin(1-y)}{1-y}\,dx\,dy$$
$$= \int_0^1 \sin(1-y)\,dy$$
$$= \cos(1-y)\Big|_0^1$$
$$= 1-\cos 1$$

Q3. Solution:

(a) For $\vec{F}$ is conservative, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

That is $ax^2+3y^2 = 6x^2+by^2$.

Hence, $a=6$, $b=3$.

(b) Now $\frac{\partial f}{\partial x} = 6x^2y+y^3+1$, so $f = 2x^3y+xy^3+x+\varphi(y)$.

And $\frac{\partial f}{\partial y} = 2x^3+bxy^2+2 = 2x^3+3xy^2+2$, so $\varphi'(y) = 2$. Thus $\varphi(y) = 2y+C$, where C is constant

Then $f = 2x^3y+xy^3+x+2y+C$, C is constant.

(c) Since $\vec{F} = \nabla f$, $\int_C \vec{F}\,d\vec{r} = f(e^t\cos t, e^t\sin t)\Big|_0^{\pi} = f(-e^{\pi},0) - f(1,0)$

$$= -e^{\pi}-1$$

Q 4. Solution: (a) Since C is a piecewise smooth, simple closed curve, by Green's Theorem:

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy$$

Now $M = xy + \sin x \cos y$, $N = -\cos x \sin y$

So $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = y$

That is $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R y\, dy\, dx = 8$

(b) We only need to compute $\oint_{C_4} \vec{F}\cdot\vec{n}\,ds$, which equals to

$$\oint_{C_4} M dy - N dx \quad (\text{Consider } C_4: x = 0,\ y = 4 - t,\ 0 \le t \le 4)$$

$$= 0 + \oint \sin(4-t)\, d0 = 0$$

Thus $\oint_{C_1+C_2+C_3} \vec{F}\cdot\vec{n}\,ds = 8 - 0 = 8$.

Q 5. Solution: We compute $V = \iint_R 4 - (2x-y)^2 - (x+y-1)^2\, dA$.

Let $u = 2x - y$, $v = x + y - 1$

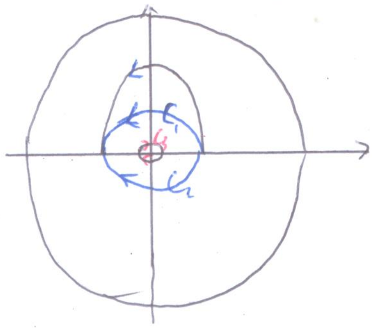
Then $x = \frac{u+v+1}{3}$, $y = \frac{-u+2v+2}{3}$

Then $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{vmatrix} = \frac{1}{3}$.

Thus $V = \iint_{u^2+v^2 \le 4} (4 - u^2 - v^2)\frac{1}{3}\, du dv = \frac{1}{3}\int_0^{2\pi}\int_0^2 (4 - r^2)\cdot r\, dr\, d\theta$

$= \frac{8}{3}\pi$.

Q6. Solution: Now $\vec{F}$ is defined in the annulus $\frac{1}{100} \le x^2+y^2 \le 100$ with $\operatorname{curl} \vec{F} = 0$.



Let C_3 be the curve $x^2+y^2 = \frac{1}{100}$ and oriented clockwisely, then By Green's Thm,

$$\int_{C_1} \vec{F}\cdot dr + \left(-\int_{C_2} \vec{F}\cdot dr\right) + \int_{C_3} \vec{F}\cdot dr = \iint \operatorname{curl} \vec{F}\, dA.$$

Thus $\int_{C_3} \vec{F}\, dr = -9$.

(a) Let C be the upper half of $x^2 + \frac{y^2}{16} = 1$ oriented from $(1,0)$ to $(-1,0)$.

Then $\int_C \vec{F}\cdot dr - \int_{C_2} \vec{F}\cdot dr + \int_{C_3} \vec{F}\cdot dr = 0 = \iint \operatorname{curl} \vec{F}\, dA$

Thus $\int_C \vec{F}\cdot dr = 5$

(b) Now $\int_C \vec{F}\cdot dr + \int_{C_3} \vec{F}\, dr = 0 = \iint \operatorname{curl} \vec{F}\, dA$.

Thus $\int_C \vec{F}\cdot dr = 9$

(c) Because the enclosed region by C does not intersect C_3, we have $\iint_C \vec{F}\cdot dr = \iint \operatorname{curl} \vec{F}\, dA = 0$.